On the Conditions to Extend Ricci Flow

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Abstract

Consider $\{(M^n, g(t)),\ 0 \leq t < T < \infty\}$ as an unnormalized Ricci flow solution: $\frac{dg_{ij}}{dt} = -2R_{ij}$ for $t \in [0, T)$. Richard Hamilton shows that if the curvature operator is uniformly bounded under the flow for all $t \in [0, T)$ then the solution can be extended over T. Natasa Sesum proves that a uniform bound of Ricci tensor is enough to extend the flow. We show that if Ricci is bounded from below, then a scalar curvature integral bound is enough to extend flow, and this integral bound condition is optimal in some sense.

1 When can Ricci flow be extended?

In (7), R. Hamilton introduces Ricci flow which deforms Riemannian metrics in the direction of the Ricci tensor. One hopes that the Ricci flow will deform any Riemannian metric to some canonical metrics, such as Einstein metrics. One can even understand geometric and topological structure of the underlying differential manifold by this sort of deformation. The idea is best illustrated in (7) where Hamilton proves that in any simply connected 3 manifold without boundary, any Riemannian metric with positive Ricci curvature can be deformed into a positive space form (up to scaling). Consequently, R. Hamilton proves that the underlying manifold is indeed diffeomorphic to S^3. This fundamental work sparks a great interest of many mathematicians in Ricci flow. In a series of work, R. Hamilton introduces an ambitious program to prove the Poincarè conjecture via Ricci flow (cf. (9) for Hamilton's program and early references in Ricci flow.). The celebrated work of G. Perelman (14), (15) and (16) indeed proves the Poincarè conjecture which states that every simply connected 3 manifold is S^3. We refer the readers to (12), (13) for more information.

After Perelman's work in the Ricci flow, there is a renewed interest in Ricci flow and its application around the world. We will refer readers to the book (4) for more updated references. In this note, we want to concentrate in studying some basic issue on Ricci flow: the maximal existence time of Ricci flow and the geometric conditions that might affect the maximal existence time.

One notes that Ricci flow is a weak Parabolic flow. R. Hamilton first proves that for any smooth initial data, the flow will exist for a short time in (7). In (6), Hamilton's proof is simplified greatly by a clever choice of gauge. The next immediate question is the so called "maximal existence time" for the Ricci flow (with respect to initial metric). In (9), Hamilton proves that if $T < \infty$ is the maximal existence time of a closed Ricci flow solution $\{(M^n, g(t)), 0 \leq t <$

$T < \infty\}$, then Riemannian curvature is unbounded as $t \to T$. In other words, a uniform bound for Riemannian curvature on $M \times [0, T)$ is enough to extend Ricci flow over time T. In (18), by a blowup argument, Sesum shows that Ricci curvature uniformly bounded on $M \times [0, T)$ is enough to extend Ricci flow over T. Sesum's surprising work uses the no local collapsing theorem of Perelman. A natural question arises: what is the optimal condition for the Ricci flow to be extended? In many ways, we believe that the scalar curvature bound shall be enough to extend the flow. In this note, we first prove (See Definition 2.1 for notations),

Theorem 1.1. *$\{(M^n, g(t)), 0 \le t < T < \infty\}$ is a closed Ricci flow solution. If*

1. *$Ric(x, t) \ge -A$ for all $(x, t) \in M \times [0, T)$, A is a positive constant ,*
2. *$\|R\|_{\alpha,\ M\times[0,T)} < \infty$, $\alpha \ge \frac{n+2}{2}$,*

then this flow can be extended over time T.

and

Theorem 1.2. *$\{(M^n, g(t)), 0 \le t < T < \infty\}$ is a closed Ricci flow solution. If*

$$\|Rm\|_{\alpha,\ M\times[0,T)} < \infty, \quad \alpha \ge \frac{n+2}{2},$$

then this flow can be extended over time T.

These two theorems are optimal in some aspects as illustrated by Example 2.1 in the next section.

Remark 1.1. *In theorem 1.1, 1.2, let $\alpha = \infty$, we can recover Sesum's and Hamilton's results.*

Organization Let's sketch the outline of this note. We first fix some notations in section 2. Then, in section 3, we prove Theorem 1.2 for all $n \ge 2$. In section 4, we use no local collapsing theorem and Croke's argument to establish a local Sobolev constant control. Then we use this control to develop a general parabolic Moser iteration under Ricci flow in section 5. Applying Moser iteration to R in section 6, we prove Theorem 1.1 for $n \ge 3$.

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2 Preliminary

Let M^n be a connected compact manifold without boundary. $(M^n, g(t))$ is called a closed Ricci flow solution if the metric satisfies the equation:

$$\frac{dg_{ij}}{dt} = -2R_{ij}. \tag{1}$$

By direct calculation, we have the evolution equations for curvatures under Ricci flow:

$$\frac{\partial R}{\partial t} = \triangle R + 2|Ric|^2, \tag{2}$$

$$\frac{\partial R_{ij}}{\partial t} = \triangle R_{ij} + 2R_{iklj}R_{kl} - 2R_{ik}R_{kj}, \tag{3}$$

$$\begin{aligned}\frac{\partial R_{ijkl}}{\partial t} &= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - (R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}),\end{aligned} \tag{4}$$

where $B_{ijkl} \triangleq -R_{ipqj}R_{kpql}$.

The evolution equation of volume element is

$$\frac{\partial d\mu}{\partial t} = -Rd\mu. \tag{5}$$

For convenience, we define some norm of the space time manifold $M \times [0, T)$ below.

Definition 2.1. *Suppose $N \subset M$, for any measurable function F defined on $N \times [0, T)$ and $\alpha \geq 1$, we define*

$$\begin{aligned}&\|F\|_{\alpha, N\times[0,T)} \triangleq (\int_0^T \int_N |F|^\alpha d\mu dt)^{\frac{1}{\alpha}}, \\ &\|F(\cdot, t)\|_{\alpha, N} \triangleq (\int_N |F|^\alpha_{g(t)} d\mu_{g(t)})^{\frac{1}{\alpha}}, \\ &F_+ \triangleq \max\{F, 0\}, \qquad F_- \triangleq \max\{-F, 0\}.\end{aligned}$$

Now we are ready to give example to illustrate that Theorem 1.1 is sharp in some aspects.

Example 2.1. *Let (S^n, g_s) be the space form of constant sectional curvature 1. Now we start Ricci flow from metric (S^n, g_s). By direct calculation, $g(t) = (1-2(n-1)t)g_s$ is the Ricci flow solution. Therefore, $T = \frac{1}{2(n-1)}$ is the maximal existence time. However, we compute*

$$\begin{aligned}\|R\|_{\alpha,\ M\times[0,T)} &= \{\int_0^T \int_M |R|^\alpha d\mu dt\}^{\frac{1}{\alpha}} \\ &= \{\int_0^T V(t)(\frac{n}{2(T-t)})^\alpha dt\}^{\frac{1}{\alpha}} \\ &= \frac{n}{2}V(0)^{\frac{1}{\alpha}} T^{-\frac{n}{2\alpha}} \{\int_0^T (T-t)^{\frac{n}{2}-\alpha} dt\}^{\frac{1}{\alpha}},\end{aligned}$$

therefore,

$$\|R\|_{\alpha,\ M\times[0,T)} \begin{cases} = \infty, & \alpha \geq \frac{n}{2}+1, \\ < \infty, & \alpha < \frac{n}{2}+1.\end{cases}$$

Moreover, $Ric \geq 0$. This suggests us that Theorem 1.1 cannot be improved to $\alpha < \frac{n+2}{2}$.

Since S^n is space form, $|Rm|^2 = C(n)^2|R|^2$, then

$$\|Rm\|_{\alpha,\ M\times[0,T)} = C(n)\|R\|_{\alpha,\ M\times[0,T)}.$$

Hence,

$$\|Rm\|_{\alpha,\ M\times[0,T)} \begin{cases} = \infty, & \alpha \geq \frac{n}{2}+1, \\ < \infty, & \alpha < \frac{n}{2}+1. \end{cases}$$

This implies Theorem 1.2 can not be improved to $\alpha < \frac{n+2}{2}$.

The uniform Sobolev constant control will play an important role in our proof.

Definition 2.2. *Suppose $\{(M^n, g(t)), 0 \leq t < T < \infty\}$ is a closed Ricci flow solution, $N \subsetneq M$. We say σ is a uniform Sobolev constant for N at each time slice, if*

$$\left(\int_N |v|_{g(t)}^{\frac{2n}{n-2}} d\mu_{g(t)}\right)^{\frac{n-2}{n}} \leq \sigma \int_N |\nabla v|_{g(t)}^2 d\mu_{g(t)}, \tag{6}$$

for every function $v \in W_0^{1,2}(N)$ and $0 \leq t < T$.

If Ricci is bounded from below, we can control $\frac{\partial R}{\partial t}$ by R.

Property 2.1. *Suppose $Ric \geq -B$, let $\hat{R} = R + nB$, then*

$$\frac{\partial \hat{R}}{\partial t} \leq \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2. \tag{7}$$

Proof. Choose an orthonormal basis to diagonalize Ric such that $Ric = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$, then

$$Ric + BI = \mathrm{diag}\{\lambda_1 + B, \cdots, \lambda_n + B\},$$

where each term is nonnegative. Therefore,

$$(\lambda_1 + B)^2 + \cdots (\lambda_n + B)^2 \leq (\lambda_1 + B + \cdots + \lambda_n + B)^2,$$

consequently,

$$\lambda_1^2 + \cdots + \lambda_n^2 \leq (\lambda_1 + \cdots + \lambda_n)^2 + 2(n-1)B(\lambda_1 + \cdots + \lambda_n) + n(n-1)B^2,$$

i.e.

$$\begin{aligned} |Ric|^2 &\leq R^2 + 2(n-1)BR + n(n-1)B^2 \\ &= \hat{R}^2 - 2B\hat{R} + nB^2. \end{aligned} \tag{8}$$

From inequality (2), we have

$$\begin{aligned} \frac{\partial \hat{R}}{\partial t} &= \frac{\partial R}{\partial t} \\ &= \triangle R + 2|Ric|^2 \\ &\leq \triangle \hat{R} + 2(\hat{R}^2 - 2B\hat{R} + nB^2) \\ &= \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2. \end{aligned}$$

□

In (14), Perelman proves the fundamental no local collapsing Theorem:

Theorem 2.1. $\{(M^n, g(t)), 0 \leq t < T < \infty\}$ *is a closed Ricci flow solution. Then there exists a* $\kappa > 0$*, such that for any* $(x,t) \in M \times [0,T), r > 0$*, if* $\sup_{y \in B_{g(t)}(x,r)} |Rm|(y,t) \leq r^{-2}$*, then*

$$\frac{\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r))}{r^n} \geq \kappa.$$

Actually, Perelman has already noticed that the same conclusion still holds if we replace the Riemannian curvature by scalar curvature. That is the next theorem.

Theorem 2.2. $\{(M^n, g(t)), 0 \leq t < T < \infty\}$ *is a closed Ricci flow solution. Then there exists a* $\kappa > 0$*, such that for any* $(x,t) \in M \times [0,T), r > 0$*, if* $\sup_{y \in B_{g(t)}(x,r)} |R(y,t)| \leq r^{-2}$*, then*

$$\frac{\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r))}{r^n} \geq \kappa.$$

The proof of Theorem 2.2 can be found in (12), (19). We will use Theorem 2.2 to get Sobolev constant control.

3 Proof of Theorem 1.2 for $n \geq 2$

Proof. By Hölder's inequality, $\|Rm\|_{\alpha, M \times [0,T)} < \infty$ implies $\|Rm\|_{\frac{n+2}{2}, M \times [0,T)} < \infty$ if $\alpha > \frac{n+2}{2}$. So we only need to prove Theorem 1.2 for $\alpha = \frac{n+2}{2}$.

We argue by contradiction.

Suppose T is the maximal existence time. Then there is a sequence $(x^{(i)}, t^{(i)})$ with $\lim_{i \to \infty} t^{(i)} = T$ and $\lim_{i \to \infty} |Rm|^{(i)} = \infty$. Moreover,

$$|Rm|(x^{(i)}, t^{(i)}) = \max_{(x,t) \in M \times [0, t^{(i)}]} |Rm|(x,t).$$

Let

$$Q^{(i)} \triangleq |Rm|(x^{(i)}, t^{(i)}),$$
$$g^{(i)}(t) \triangleq Q^{(i)} g((Q^{(i)})^{-1} t + t^{(i)}).$$

By Theorem 2.1, we have uniform lower bound of injectivity radius at points $(x^{(i)}, t^{(i)})$ for the sequence $\{((M^n, x^{(i)}), g^{(i)}(t)),\ -Q^{(i)} t^{(i)} \leq t \leq 0\}$. So it subconverges to an ancient Ricci flow solution $\{((\bar{M}, \bar{x}), \bar{g}(t)), -\infty \leq t \leq 0\}$. Therefore, by the scaling invariance of $\int_0^T \int_M |Rm|^{\frac{n+2}{2}} d\mu dt$, we have

$$\begin{aligned}
\int_{-1}^{0} \int_{B_{\bar{g}(0)}(\bar{x},1)} |\bar{Rm}|^{\frac{n+2}{2}} d\bar{\mu} dt &\leq \lim_{i \to \infty} \int_{-1}^{0} \int_{B_{g^{(i)}(0)}(x^{(i)},1)} |Rm|^{\frac{n+2}{2}}_{g^{(i)}(t)} d\mu_{g^{(i)}(t)} dt \\
&= \lim_{i \to \infty} \int_{t^{(i)} - (Q^{(i)})^{-1}}^{t^{(i)}} \int_{B_{g(t^{(i)})}(x^{(i)}, (Q^{(i)})^{-\frac{1}{2}})} |Rm|^{\frac{n+2}{2}} d\mu dt \\
&\leq \lim_{i \to \infty} \int_{t^{(i)} - (Q^{(i)})^{-1}}^{t^{(i)}} \int_M |Rm|^{\frac{n+2}{2}} d\mu dt \\
&= 0. \qquad (9)
\end{aligned}$$

The last equality holds since $\int_0^T \int_M |Rm|^{\frac{n+2}{2}} d\mu dt < \infty$ and $\lim_{i\to\infty}(Q^{(i)})^{-\frac{1}{2}} = 0$. Since $(\bar{M}, \bar{g}(t))$ is a smooth Riemannian manifold for each $t \leq 0$, equality (9) implies that $|\bar{Rm}| \equiv 0$ on the parabolic ball $B_{\bar{g}(0)(\bar{x},1)} \times [-1, 0]$. In particular, $|\bar{Rm}|(\bar{x}, 0) = 0$. On the other hand,

$$|\bar{Rm}|(\bar{x}, 0) = \lim_{i\to\infty} |Rm|_{g^{(i)}}(x^{(i)}, 0) = 1.$$

So we get a contradiction. $\square$

When dimension is 2, $Rm = R$. Thus Theorem 1.1 and Theorem 1.2 are the same. So we have already proved Theorem 1.1 for $n = 2$. When $n \geq 3$, R and Rm are different. Accordingly we have to develop some new techniques to prove Theorem 1.1. Moser iteration will play a critical role in our proof. In order to apply Moser iteration, we need to get a local Sobolev constant control first.

4 Local Sobolev Constant Control

In this section, we discuss how to control isoperimetric constant locally. By the equivalence of isoperimetric constant and Sobolev constant, we get the local control for Sobolev constant. The following argument comes from Croke's paper (5).

Definition 4.1. *Suppose $(N, \partial N, g)$ be a smooth compact manifold with smooth boundary and Riemannian metric g.*

$$\Phi(N) \triangleq \inf_{\Omega \Subset N} \frac{\mathrm{Area}(\partial\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}}.$$

Let $UN \xrightarrow{\pi} N$ represent the unit sphere bundle with the canonical measure. For $v \in UN$, let γ_v be the geodesic with $\gamma_v'(0) = v$, let $\zeta^t(v)$ represent the geodesic flow, i.e. $\zeta^t(v) = \gamma_v'(t)$. Let $l(v)$ be the smallest value of $t > 0$ (possibly ∞) such that $\gamma_v(t) \in \partial N$. Note $\zeta^t(v)$ is defined for $t \leq l(v)$. Let

$$\begin{aligned}
\tilde{l}(v) &\triangleq \sup\{t | \gamma_v \textit{ minimizes up to } t \textit{ and } t \leq l(v)\}, \\
\tilde{U}M &\triangleq \{v \in UM | \tilde{l}(v) = l(v)\}, \qquad \tilde{U}_p \triangleq \pi|_{\tilde{U}M}^{-1}(p), \\
\tilde{\omega}_p &\triangleq \frac{\mathrm{Area}\, \tilde{U}_p}{\mathrm{Area}\, U_p}, \qquad \tilde{\omega} \triangleq \inf_{p\in N} \tilde{\omega}_p, \\
\alpha(n) &\triangleq \textit{ volume of unit sphere of dimension } n.
\end{aligned}$$

For $p \in \partial N$, let N_p be the inwardly pointing unit normal vector. Let $U^+\partial N \to \partial N$ be the bundle of inwardly pointing unit vectors. That is,

$$U^+\partial N = \{u \in UN|_{\partial N} | \langle u, N_{\pi(u)} \rangle \geq 0\}.$$

$U^+\partial N$ has natural metric structure.

This $\tilde{\omega}$ is related to $\Phi(N)$ closely. If we have a control over $\tilde{\omega}$, then it's easy to get a control for $\Phi(N)$.

Proposition 4.1. *For $(N, \partial N, g)$ we have*

$$\int_{\tilde{U}N} f(v)dv = \int_{U^+\partial N} \int_0^{\tilde{l}(u)} f(\zeta^r(u)) < u, N_{\pi(u)} > drdu, \tag{10}$$

where f is any integrable function. In particular for $f \equiv 1$, we have

$$\mathrm{Vol}(\tilde{U}M) = \int_{U^+\partial N} \tilde{l}(u)\langle u, N_{\pi(u)}\rangle du. \tag{11}$$

This formula occurs in (1), p.286, and (17), pp.336-338.

Proposition 4.2. *Let N^n be a Riemannian manifold and $u \in UN$. Then for every $l \leq C(u)$ (the distance to the cut locus in the direction u):*

$$\int_{x=0}^{x=l} \int_{z=0}^{z=l-x} F(\zeta^x(u), z)dzdx \geq C_1(n) \cdot \frac{l^{n+1}}{\pi^{n+1}}, \tag{12}$$

where $C_1(n) = \frac{\pi\alpha(n)}{2\alpha(n-1)}$. $F(v, z)$ is the volume form in polar coordinates,i.e.,

$$\int_{U_p} \int_0^{C(v)} F(v, z)dzdv = \mathrm{Vol}(M).$$

The proof can be found in Berger's work (2) (Appendix D).

Lemma 4.1. *For $(N, \partial N, g)$ we have the isoperimetric inequality:*

$$\frac{\mathrm{Area}(\partial N)^n}{\mathrm{Vol}(N)^{n-1}} \geq C_2(n)\tilde{\omega}^{n+1}, \tag{13}$$

where $C_2(n) = 2^{n-1}\frac{\alpha(n-1)^n}{\alpha(n)^{n-1}}$.

Proof.

$$\begin{aligned}
\mathrm{Vol}(N)^2 &= \int_N \mathrm{Vol}(N)dp \\
&\geq \int_N \int_{U_p} \int_0^{\tilde{l}(u)} F(u, t)dtdudp \\
&= \int_{UN} \int_0^{\tilde{l}(u)} F(u, t)dtdu \\
&\geq \int_{\tilde{U}N} \int_0^{\tilde{l}(u)} F(u, t)dtdu \\
&= \int_{U^+\partial N} \int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(\zeta^s(v))} F(\zeta^s(v), t)\langle v, N_{\pi(v)}\rangle dtdsdv \\
&= \int_{U^+\partial N} [\int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(v)-s} F(\zeta^s(v), t)dtds]\langle v, N_{\pi(v)}\rangle dv \\
&\geq \frac{C_1(n)}{\pi^{n+1}} \int_{U^+\partial N} (\tilde{l}(v))^{n+1}\langle v, N_{\pi(v)}\rangle dv.
\end{aligned} \tag{14}$$

By Hölder inequality,

$$\int_{U^+\partial N}\tilde{l}(v)\langle v,N_{\pi(v)}\rangle dv=\int_{U^+\partial N}(\tilde{l}(v)\langle v,N_{\pi(v)}\rangle^{\frac{1}{n+1}})\langle v,N_{\pi(v)}\rangle^{\frac{n}{n+1}}dv$$
$$\leq(\int_{U^+\partial N}\tilde{l}^{n+1}\langle v,N_{\pi(v)}\rangle dv)^{\frac{1}{n+1}}(\int_{U^+\partial N}\langle v,N_{\pi(v)}\rangle dv)^{\frac{n}{n+1}},$$

then,

$$\int_{U^+\partial N}\tilde{l}^{n+1}\langle v,N_{\pi(v)}\rangle dv\geq\frac{(\int_{U^+\partial N}\tilde{l}(v)\langle v,N_{\pi(v)}\rangle dv)^{n+1}}{(\int_{U^+\partial N}\langle v,N_{\pi(v)}\rangle dv)^n}. \tag{15}$$

Put inequality (15) into inequality (14), we get

$$\mathrm{Vol}(N)^2\geq\frac{C_1(n)}{\pi^{n+1}}\frac{(\int_{U^+\partial N}\tilde{l}(v)\langle v,N_{\pi(v)}\rangle dv)^{n+1}}{(\int_{U^+\partial N}\langle v,N_{\pi(v)}\rangle dv)^n},$$

therefore,

$$\mathrm{Vol}(N)^2(\int_{U^+\partial N}\langle v,N_{\pi(v)}\rangle dv)^n\geq\frac{C_1(n)}{\pi^{n+1}}\mathrm{Vol}(\tilde{U}M)^{n+1}$$
$$\geq\frac{C_1(n)}{\pi^{n+1}}[\tilde{\omega}\alpha(n-1)\mathrm{Vol}(N)]^{n+1}.$$

Note that

$$\int_{U^+\partial N}\langle v,N_{\pi(v)}\rangle dv=\frac{\alpha(n)}{2\pi}\mathrm{Area}(\partial N),$$

consequently,

$$\begin{aligned}\frac{\mathrm{Area}(\partial N)^n}{\mathrm{Vol}(N)^{n-1}}&\geq\frac{C_1(n)}{\pi^{n+1}}\tilde{\omega}^{n+1}\alpha(n-1)^{n+1}\frac{(2\pi)^n}{\alpha(n)^n}\\&=2^{n-1}\frac{\alpha(n-1)^n}{\alpha(n)^{n-1}}\tilde{\omega}^{n+1}\\&\triangleq C_2(n)\tilde{\omega}^{n+1}.\end{aligned}$$

□

Lemma 4.2. *M is a complete Riemannian manifold with $Ric\geq-(n-1)K^2$. $\Omega\Subset N_1\subset N_2\subset M$, Ω is a domain with smooth boundary, and* $\mathrm{diam}(N_2)\leq D$. *Then*

$$\tilde{\omega}(\Omega)\geq\frac{\mathrm{Vol}(N_2)-\mathrm{Vol}(N_1)}{\alpha(n-1)\int_0^D(\frac{\sinh Kr}{K})^{n-1}dr}. \tag{16}$$

Proof. Choose $p\in\Omega$. Then $(\Omega,\partial\Omega,g)$ is a smooth Riemannian manifold with boundary. We look $(\Omega,\partial\Omega,g)$ as $(N,\partial N,g)$ in our previous argument. Let

$$O_p\triangleq\{q\in M|q=\exp_p tu,u\in\tilde{U}_p\ ,t\leq C(u)\},$$

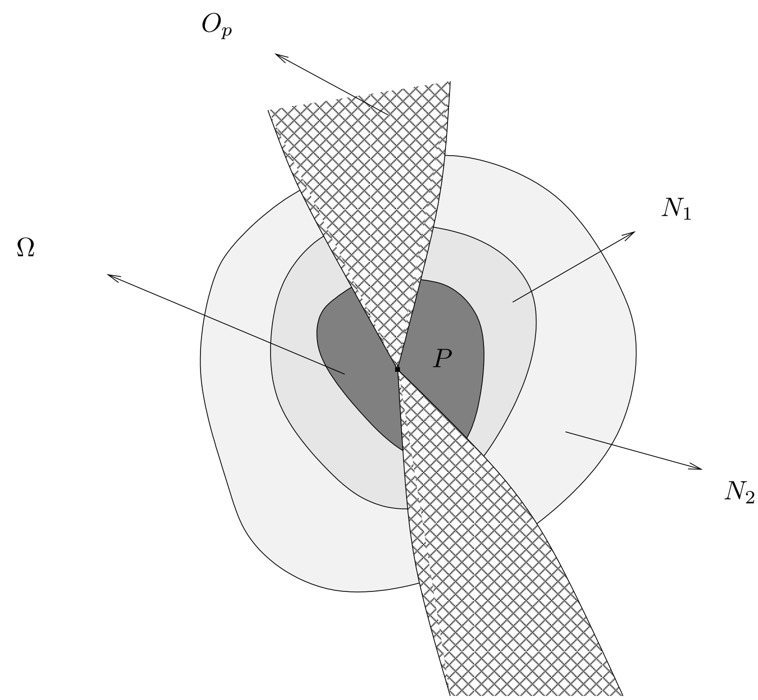

Figure 1: the relation among sets

where $C(u)$ is the cut radius at direction u. Since $u \in \tilde{U}_p$, $\tilde{l}(u) = l(u)$. Therefore $M \setminus \Omega \subset O_p$, in particular, $N_2 \setminus N_1 \subset O_p$. And also we know, $N_2 \setminus N_1 \subset N_2 \subset B(p, D)$. Then

$$\begin{aligned} \mathrm{Vol}(N_2 \setminus N_1) &\leq \mathrm{Vol}(O_p \cap B(p, D)) \\ &= \int_{\tilde{U}_p} \int_0^D F(u, r) dr du \\ &\leq \tilde{\omega}_p \alpha(n-1) \int_0^D (\frac{\sinh Kr}{K})^{n-1} dr. \end{aligned}$$

Consequently,

$$\tilde{\omega}_p \geq \frac{\mathrm{Vol}(N_2) - \mathrm{Vol}(N_1)}{\alpha(n-1) \int_0^D (\frac{\sinh Kr}{K})^{n-1} dr}.$$

Since p is an arbitrary point in Ω, we have

$$\tilde{\omega} = \inf_{p \in \Omega} \tilde{\omega}_p \geq \frac{\mathrm{Vol}(N_2) - \mathrm{Vol}(N_1)}{\alpha(n-1) \int_0^D (\frac{\sinh Kr}{K})^{n-1} dr}.$$

□

Theorem 4.1. *Suppose* $\{(M^n, g(t)), 0 \leq t \leq 1\}, n \geq 3$ *is a Ricci flow solution.*

$p \in M$, and

$$\begin{aligned}
&Ric(x,t) \geq -(n-1),\ \forall\ (x,t) \in M \times [0,1];\\
&Ric(x,t) \leq (n-1),\ \forall\ (x,t) \in B_{g(1)}(p,1) \times [0,1];\\
&\mathrm{Vol}_{g(1)}(B_{g(1)}(p,1)) \geq \kappa.
\end{aligned}$$

Let $r(\kappa)$ be the solution of $\int_0^{r(\kappa)} (\sinh s)^{n-1} ds = \frac{\kappa}{2\alpha(n-1)e^{2n(n-1)}}$. Then there is a uniform Sobolev constant $\sigma(n,\kappa)$ for $B_{g(1)}(p,r(\kappa))$ on each time slice, i.e., for any $f \in W_0^{1,2}(B_{g(1)}(p,r(\kappa)))$,

$$\|f\|^2_{\frac{2n}{n-2},B_{g(1)}(p,r(\kappa))} \leq \sigma(n,\kappa)\|\nabla f\|^2_{2,B_{g(1)}(p,r(\kappa))}. \tag{17}$$

Proof. Let $N_1 \triangleq B_{g(1)}(p,r(\kappa))$, $N_2 \triangleq B_{g(1)}(p,1)$. Calculating the evolution equation for volume:

$$\begin{aligned}
\frac{d\,\mathrm{Vol}_{g(t)}(N_2)}{dt} &= -\int_{N_2} R d\mu\\
&\leq n(n-1)\,\mathrm{Vol}_{g(t)}(N_2), \quad (Ric \geq -(n-1))
\end{aligned}$$

hence,

$$\begin{aligned}
\mathrm{Vol}_{g(t)}(N_2) &\geq e^{n(n-1)(t-1)}\,\mathrm{Vol}_{g(1)}(N_2)\\
&\geq e^{-n(n-1)}\,\mathrm{Vol}_{g(1)}(N_2) \quad (0 \leq t \leq 1)\\
&\geq e^{-n(n-1)}\kappa.
\end{aligned} \tag{18}$$

Similarly, by the condition $Ric \leq (n-1)$,

$$\begin{aligned}
\mathrm{Vol}_{g(t)}(N_1) &\leq e^{n(n-1)(1-t)}\,\mathrm{Vol}_{g(1)}(N_1)\\
&\leq e^{n(n-1)}\,\mathrm{Vol}_{g(1)}(N_1)\\
&= e^{n(n-1)} \int_{B_{g(1)}(p,r(\kappa))} d\mu\\
&\leq e^{n(n-1)}\alpha(n-1)\int_0^{r(\kappa)} (\sinh s)^{n-1} ds\\
&\leq \frac{\kappa}{2} e^{-n(n-1)}.
\end{aligned} \tag{19}$$

Now we consider the diameter change under Ricci flow. Suppose $\{\gamma(s), 0 \leq s \leq \rho\}$ is a normalized shortest geodesic contained in N_2 at time t, then

$$\begin{aligned}
\frac{dL_{g(t)}(\gamma)}{dt} &= -\int_0^\rho Ric(\gamma',\gamma')ds\\
&\geq -(n-1)L_{g(t)}(\gamma).
\end{aligned}$$

Let $D(t)$ be the diameter of N_2 at time t, we have

$$\frac{d^- D(t)}{dt} \geq -(n-1)D(t),$$

hence,

$$D(t) \leq D(1)e^{(n-1)(1-t)} \leq 2e^{(n-1)}. \tag{20}$$

Choose an arbitrary domain $\Omega \Subset N_1$ with smooth boundary.By inequalities (18),(19) and (20), from lemma 4.2, we know

$$\begin{aligned}\tilde{\omega}_{g(t)}(\Omega) &\geq \frac{\mathrm{Vol}_{g(t)}(N_2) - \mathrm{Vol}_{g(t)}(N_1)}{\alpha(n-1)\int_0^{D(t)}(\sinh s)^{n-1}ds} \\ &\geq \frac{\kappa e^{-n(n-1)}}{2\alpha(n-1)\int_0^{2e^{(n-1)}}(\sinh s)^{n-1}ds} \\ &\triangleq C_3(n,\kappa).\end{aligned}$$

Then, from lemma 4.1, we have

$$\begin{aligned}\frac{\mathrm{Area}(\partial\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} &\geq C_2(n)C_3(n,\kappa)^{n+1} \\ &\triangleq C_4(n,\kappa).\end{aligned}$$

Since we can approximate any domain by domains with smooth boundary, we actually get

$$\Phi(N_1) = \inf_{\Omega \Subset N_1} \frac{\mathrm{Area}(\partial\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geq C_4(n,\kappa). \tag{21}$$

Accordingly, by the equivalence of isoperimetric constant and Sobolev constant, for any $f \in W_0^{1,1}(N_1)$,

$$C_4(n,\kappa)(\int_{N_1}|f|^{\frac{n}{n-1}}d\mu)^{\frac{n-1}{n}} \leq \int_{N_1}|\nabla f|. \tag{22}$$

We refer the readers to (20) for a detailed proof for the equivalence of inequality (21) and inequality (22). Let $\gamma > 0$, then

$$\begin{aligned}\||f|^\gamma\|_{\frac{n}{n-1},N_1} &\leq \frac{1}{C_4}\|\gamma|f|^{\gamma-1}\nabla f\|_{1,N_1} \\ &\leq \frac{1}{C_4}\gamma\|f^{\gamma-1}\|_{\frac{p}{p-1},N_1}\|\nabla f\|_{p,N_1}\ ,\end{aligned}$$

therefore,

$$\|f\|^\gamma_{\frac{n\gamma}{n-1},N_1} \leq \frac{1}{C_4}\|f\|^{\gamma-1}_{\frac{(\gamma-1)p}{p-1},N_1} \cdot \|\nabla f\|_{p,N_1}\ .$$

Choose $\gamma = \frac{p(n-1)}{n-p}$, we have

$$\|f\|_{\frac{np}{n-p},N_1} \leq \frac{1}{C_4} \cdot \frac{p(n-1)}{n-p} \cdot \|\nabla f\|_{p,N_1}\ .$$

In particular, choose $p = 2$, let

$$\sigma(n,\kappa) = (\frac{2(n-1)}{C_4(n,\kappa)(n-2)})^2,$$

we obtain

$$\|f\|^2_{\frac{2n}{n-2},N_1} \leq \sigma(n,\kappa)\|\nabla f\|^2_{2,N_1}$$

for any $f \in W^{1,2}_0(N_1)$. $\square$

After we get the local Sobolev constant control, we are able to get some Moser iteration formula under Ricci flow.

5 Moser Iteration of Scalar curvature ($n \geq 3$)

We will give a detailed construction of local Moser iteration under Ricci flow in this section. The idea comes from the Moser iteration in (3). Let us fix notation first.

Definition 5.1. $\{(M^n, g(t)),\ 0 \leq t \leq 1\}$ *is a closed Ricci flow solution. Fixing* $p \in M$, $r > 0$, *we define*

$$\Omega \triangleq B_{g(1)}(p,r), \qquad \Omega' \triangleq B_{g(1)}(p,\frac{r}{2}),$$

$$D \triangleq \Omega \times [0,1], \qquad D' \triangleq \Omega' \times [\frac{1}{2},1].$$

Inequality (6) is only Sobolev inequality for time slices. In order to apply Moser iteration on the parabolic domain D, we need a parabolic version of Sobolev inequality.

Property 5.1. *Suppose there is a uniform Soblev constant σ for Ω at each time slice, $v \in C^1(D)$, and $v(\cdot,t) \in C^1_0(D)$, $\forall t \in [0,1]$, we have*

$$\int_D v^{\frac{2(n+2)}{n}} d\mu dt \leq \sigma \max_{0\leq t\leq 1} \|v(\cdot,t)\|^{\frac{4}{n}}_{2,\Omega} \int_D |\nabla v|^2 d\mu dt. \tag{23}$$

Proof. By Hölder inequality and inequality (6), we have

$$\begin{aligned}
\int_D v^{\frac{2(n+2)}{n}} d\mu dt &= \int_0^1 dt \int_\Omega v^2 \cdot v^{\frac{4}{n}} d\mu \\
&\leq \int_0^1 dt (\int_\Omega v^{\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} \cdot (\int_\Omega v^{\frac{4}{n}\cdot\frac{n}{2}} d\mu)^{\frac{2}{n}} \\
&= \int_0^1 \|v(\cdot,t)\|^{\frac{4}{n}}_{2,\Omega} (\int_\Omega v^{\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} dt \\
&\leq \sigma \max_{0\leq t\leq 1} \|v(\cdot,t)\|^{\frac{4}{n}}_{2,\Omega} \int_D |\nabla v|^2 d\mu dt.
\end{aligned}$$

$\square$

Then we start the main Lemmas in this section.

Lemma 5.1. $\{(M^n, g(t)),\ 0 \leq t \leq 1\}$ *is a closed Ricci flow solution with $Ric \geq -B$. Suppose there is a uniform Soblev constant σ for Ω at each time slice. If $u \in C^1(D)$ and $u \geq 0$,*

$$\frac{\partial u}{\partial t} \leq \triangle u + fu + h, \tag{24}$$

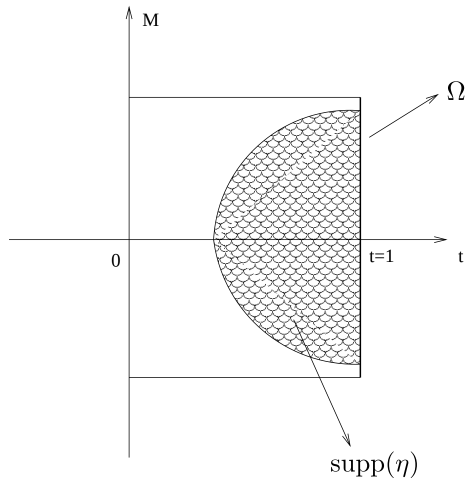

Figure 2: a cutoff function

in distribution sense, and $\|f\|_{q,D} + \|R_-\|_{q,D} + 1 \leq C_0$ *for some* $q > \frac{n}{2} + 1$*. Then there is a constant* $C_a = C_a(n, q, \sigma, C_0, r, B)$ *such that*

$$\|u\|_{\infty,D'} \leq C_a(\|u\|_{\frac{n+2}{n},D} + \|h\|_{q,D} \cdot \|1\|_{\frac{n+2}{n},D}). \tag{25}$$

Proof. Choose a cutoff function $\eta \in C^\infty(D)$ such that $\eta(\cdot, t) \in C_0^\infty(\Omega)$, $\forall t \in [0,1]$, and $\eta(x, 0) \equiv 0$. Moreover, $\eta(x, \cdot)$ is a nondecreasing function for every $x \in \Omega$.

Define

$$\kappa \triangleq \|h\|_{q,D}, \quad v \triangleq u + \kappa.$$

Fix $\beta > 1$, use $\eta^2(u+\kappa)^{\beta-1}$ as a test function, from inequality (24),

$$-\triangle v + \frac{\partial v}{\partial t} \leq fu + h.$$

Then, for any $s \in (0, 1]$,we have

$$\begin{aligned}
&\int_0^s \int_\Omega (-\triangle v)\eta^2 v^{\beta-1} d\mu dt + \int_0^s \int_\Omega \frac{\partial v}{\partial t}\eta^2 v^{\beta-1} d\mu dt \\
\leq &\int_0^s \int_\Omega (fu + h)\eta^2 (u+\kappa)^{\beta-1} d\mu dt \\
\leq &\int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa})\eta^2 v^\beta d\mu dt.
\end{aligned}$$

Note that $\frac{\partial d\mu}{\partial t} = -Rd\mu$, integrating by parts yields

$$\begin{aligned}
&\int_0^s \int_\Omega (2\eta < \nabla\eta, \nabla v > v^{\beta-1} + (\beta-1)\eta^2 v^{\beta-2}|\nabla v|^2) d\mu dt \\
&\quad + \frac{1}{\beta}(\int_\Omega \eta^2 v^\beta d\mu|_s - \int_0^s \int_\Omega 2\eta \frac{\partial\eta}{\partial t} v^\beta d\mu dt + \int_0^s \int_\Omega \eta^2 v^\beta R d\mu dt) \\
&\leq \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa})\eta^2 v^\beta d\mu dt. \qquad (26)
\end{aligned}$$

By Schwartz inequality,

$$\int_0^s \int_\Omega 2\eta < \nabla\eta, \nabla v > v^{\beta-1} d\mu dt \geq -\epsilon^2 \int_0^s \int_\Omega \eta^2 v^{\beta-2}|\nabla v|^2 - \frac{1}{\epsilon^2}\int_0^s \int_\Omega v^\beta |\nabla\eta|^2. \qquad (27)$$

Plugging inequality (27) into (26), we get

$$\begin{aligned}
&(\beta - 1 - \epsilon^2)\int_0^s \int_\Omega \eta^2 v^{\beta-2}|\nabla v|^2 d\mu dt + \frac{1}{\beta}\int_\Omega \eta^2 v^\beta d\mu|_s \\
&\leq \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa})\eta^2 v^\beta d\mu dt + \frac{1}{\epsilon^2}\int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt \\
&\quad + \frac{1}{\beta}(\int_0^s \int_\Omega 2\eta \frac{\partial\eta}{\partial t} v^\beta d\mu dt - \int_0^s \int_\Omega \eta^2 v^\beta R d\mu dt).
\end{aligned}$$

Let $\epsilon^2 = \frac{\beta-1}{2}$, since $|\nabla v^{\frac{\beta}{2}}|^2 = \frac{\beta^2}{4} v^{\beta-2}|\nabla v|^2$, we know

$$\begin{aligned}
&2(1 - \frac{1}{\beta})\int_0^s \int_\Omega \eta^2 |\nabla v^{\frac{\beta}{2}}|^2 d\mu dt + \int_\Omega \eta^2 v^\beta d\mu|_s \\
&\leq \beta \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa} + R_-)\eta^2 v^\beta d\mu dt \\
&\quad + \frac{2\beta}{\beta-1}\int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial\eta}{\partial t} v^\beta d\mu dt.
\end{aligned}$$

Since

$$|\nabla(\eta v^{\frac{\beta}{2}})|^2 \leq 2\eta^2 |\nabla v^{\frac{\beta}{2}}|^2 + 2v^\beta |\nabla\eta|^2,$$

we have

$$\begin{aligned}
&(1 - \frac{1}{\beta})\int_0^s \int_\Omega |\nabla(\eta v^{\frac{\beta}{2}})|^2 d\mu dt + \int_\Omega \eta^2 v^\beta d\mu|_s \\
&\leq \beta \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa} + R_-)\eta^2 v^\beta d\mu dt \\
&\quad + 2(\frac{\beta}{\beta-1} + \frac{\beta-1}{\beta})\int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial\eta}{\partial t} v^\beta d\mu dt.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_0^s \int_\Omega |\nabla(\eta v^{\frac{\beta}{2}})|^2 d\mu dt + \int_\Omega \eta^2 v^\beta d\mu|_s \\
&\quad \leq \Lambda(\beta)(\int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa} + R_-)\eta^2 v^\beta d\mu dt + \int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta d\mu dt) \\
&\quad \leq \Lambda(\beta)((\int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa} + R_-)^q d\mu dt)^{\frac{1}{q}} (\int_0^s \int_\Omega (\eta^2 v^\beta)^{\frac{q}{q-1}} d\mu dt)^{\frac{q-1}{q}} \\
&\qquad + \int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta d\mu dt) \\
&\quad \leq \Lambda(\beta)\{(\|f\|_{q,D} + \|R_-\|_{q,D} + 1)(\int_0^s \int_\Omega (\eta^2 v^\beta)^{\frac{q}{q-1}} d\mu dt)^{\frac{q-1}{q}} \\
&\qquad + \int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta d\mu dt\} \\
&\quad \leq \Lambda(\beta)(C_0(\int_0^s \int_\Omega (\eta^2 v^\beta)^{\frac{q}{q-1}} d\mu dt)^{\frac{q-1}{q}} + \int_0^s \int_\Omega v^\beta |\nabla\eta|^2 d\mu dt + \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta d\mu dt).
\end{aligned}
$$

We can choose $\Lambda(\beta) = 6\beta$ if $\beta \geq 2$. In particular,

$$
\begin{aligned}
&\max_{0\leq s\leq 1} \int_\Omega \eta^2 v^\beta d\mu|_s \leq \Lambda(\beta)(\|(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t})v^\beta\|_{1,D} + C_0\|\eta^2 v^\beta\|_{\frac{q}{q-1},D}), \\
&\int_D \eta^2 |\nabla v^{\frac{\beta}{2}}|^2 d\mu dt \leq \Lambda(\beta)(\|(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t})v^\beta\|_{1,D} + C_0\|\eta^2 v^\beta\|_{\frac{q}{q-1},D}).
\end{aligned}
$$

The Sobolev inequality (23) on the parabolic domain D yields

$$
\|\eta^2 v^\beta\|_{\frac{n+2}{n},D} \leq \sigma^{\frac{n}{n+2}}\Lambda(\beta)(\|(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t})v^\beta\|_{1,D} + C_0\|\eta^2 v^\beta\|_{\frac{q}{q-1},D}). \tag{28}
$$

Since $q > \frac{n+2}{2}$, $\frac{q}{q-1} \leq \frac{n+2}{n}$, by interpolation inequality,

$$
\|\eta^2 v^\beta\|_{\frac{q}{q-1},D} \leq \epsilon'\|\eta^2 v^\beta\|_{\frac{n+2}{n},D} + (\epsilon')^{-\nu}\|\eta^2 v^\beta\|_{1,D},
$$

where $\nu = \frac{n+2}{2q-n-2}$. Therefore,

$$
\begin{aligned}
&(1 - \Lambda(\beta)\sigma^{\frac{n}{n+2}}C_0\epsilon')\|\eta^2 v^\beta\|_{\frac{n+2}{n},D} \\
&\qquad \leq \Lambda(\beta)\sigma^{\frac{n}{n+2}}(\|(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t})v^\beta\|_{1,D} + C_0\cdot(\epsilon')^{-\nu}\|\eta^2 v^\beta\|_{1,D}).
\end{aligned}
$$

Let $\epsilon' = \frac{1}{2\Lambda(\beta)\sigma^{\frac{n}{n+2}}C_0}$, we get

$$
\|\eta^2 v^\beta\|_{\frac{n+2}{n},D} \leq 2\Lambda(\beta)\sigma^{\frac{n}{n+2}}(\|(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t})v^\beta\|_{1,D} + C_0\cdot(2\Lambda(\beta)\sigma^{\frac{n}{n+2}}C_0)^\nu\|\eta^2 v^\beta\|_{1,D}).
$$

Since we can always choose $\Lambda(\beta) \geq 1$, we obtain

$$
\|\eta^2 v^\beta\|_{\frac{n+2}{n},D} \leq C_1(n,q,\sigma,C_0)\Lambda(\beta)^{1+\nu}\int_D (|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t} + \eta^2)v^\beta d\mu dt. \tag{29}
$$

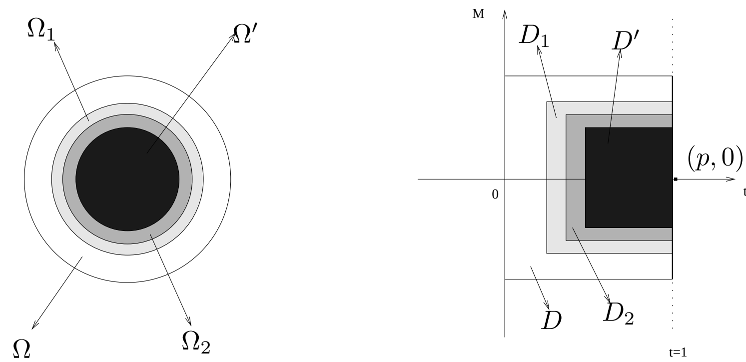

Figure 3: the sequence of domains

Then we construct cutoff functions and domains. Define

$$t_k \triangleq \frac{1}{2} - \frac{1}{2^{k+1}}, \quad r_k \triangleq (\frac{1}{2} + \frac{1}{2^{k+1}})r, \quad k \geq 0,$$

$$\Omega_k \triangleq B_{g(1)}(p, r_k), D_k \triangleq \Omega_k \times [t_k, 1], \quad k \geq 0. \tag{30}$$

Let $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$, $0 \leq \gamma' \leq 2$, and

$$\gamma(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

Define $\gamma_k(t) \triangleq \gamma(\frac{t - t_{k-1}}{t_k - t_{k-1}}), \quad k \geq 1$.

Let $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$, $-2 \leq \rho' \leq 0$, and

$$\rho(s) = \begin{cases} 1, & s \leq 0, \\ 0, & s \geq 1. \end{cases}$$

Define $\rho_k(s) \triangleq \rho(\frac{s - r_k}{r_{k-1} - r_k}), \quad k \geq 1$. Then let

$$\eta_k(x, t) = \gamma_k(t)\rho_k(d_{g(1)}(x, p)).$$

Therefore, $0 \leq \eta_k \leq 1$, and

$$\eta_k(x, t) = \begin{cases} 0, & (x, t) \in D/D_{k-1}, \\ 1, & (x, t) \in D_k. \end{cases}$$

Moreover,

$$|\frac{\partial \eta_k}{\partial t}| = |\frac{\partial \gamma_k(t)}{\partial t}\rho_k(r(x))| = |\frac{\gamma'}{t_k - t_{k-1}}\rho_k(d_{g(1)}(x, p))| \leq 2^{k+2},$$

$$\begin{aligned} |\nabla \eta_k|_{g(1)} &= |\gamma_k(t)\nabla \rho_k(d_{g(1)}(x, p))|_{g(1)} \\ &= |\gamma_k(t)\rho_k'(d_{g(1)}(r, p))\nabla d_{g(1)}(x, p)|_{g(1)} \\ &\leq |\rho_k'(d_{g(1)}(x, p))|_{g(1)} \\ &\leq \frac{\rho'}{r_{k-1} - r_k} \leq 2^{k+2} r^{-1}. \end{aligned}$$

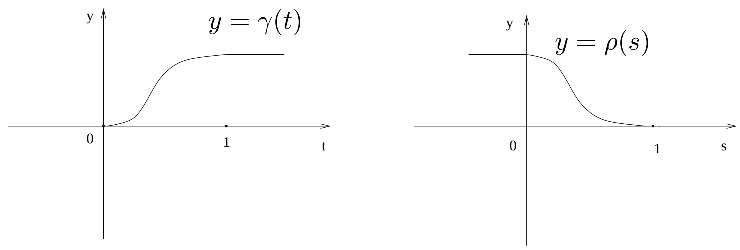

Figure 4: basic cutoff functions

Note that

$$\frac{d}{dt}|\nabla\eta_k|^2_{g(t)} = 2Ric_{g(t)}(\nabla\eta_k,\nabla\eta_k) \geq -2B|\nabla\eta_k|^2_{g(t)},$$

hence

$$|\nabla\eta_k|^2_{g(t)} \leq e^{2B(1-t)}|\nabla\eta_k|^2_{g(1)} \leq e^{2B}|\nabla\eta_k|^2_{g(1)}.$$

Therefore,we know

$$|\frac{\partial\eta_k}{\partial t}| \leq 2^{k+2},$$

$$|\nabla\eta_k|_{g(t)} \leq e^B 2^{k+2} r^{-1}, \quad \forall t\in[0,1]. \tag{31}$$

If $\beta\geq 2$,$\Lambda(\beta)=6\beta$, by inequality (29), we have

$$\begin{aligned}
\|v^\beta\|_{\frac{n+2}{n},D_k} &= \|{\eta_k}^2 v^\beta\|_{\frac{n+2}{n},D_k}\\
&\leq \|{\eta_k}^2 v^\beta\|_{\frac{n+2}{n},D_{k-1}}\\
&\leq C_2(n,q,\sigma,C_0)\beta^{1+\nu}\int_{D_{k-1}}(|\nabla\eta_k|^2+2\eta_k\frac{\partial\eta_k}{\partial t}+\eta_k^2)v^\beta d\mu dt\\
&\leq 4^{k+2}C_3(r,B)C_2(n,q,\sigma,C_0)\beta^{1+\nu}\int_{D_{k-1}}v^\beta d\mu dt\\
&\triangleq C_4(n,q,\sigma,C_0,r,B)\cdot 4^{k-1}\cdot\beta^{1+\nu}\|v^\beta\|_{1,D_{k-1}},
\end{aligned}$$

consequently,

$$\|v\|_{\frac{n+2}{n}\beta,D_k} \leq C_4^{\frac{1}{\beta}}\cdot 4^{\frac{k-1}{\beta}}\cdot\beta^{\frac{1+\nu}{\beta}}\|v\|_{\beta,D_{k-1}}\ . \tag{32}$$

Let $\lambda\triangleq\frac{n+2}{n}$, then

$$\begin{aligned}
\|v\|_{\lambda^k,D_k} &\leq C_4^{\frac{1}{\lambda^{k-1}}+\frac{1}{\lambda^{k-2}}+\cdots+\frac{1}{\lambda^{k_0}}}\, 4^{\frac{k-1}{\lambda^{k-1}}+\cdots+\frac{k_0}{\lambda^{k_0}}}\,\lambda^{(1+\nu)(\frac{k-1}{\lambda^{k-1}}+\cdots+\frac{k_0}{\lambda^{k_0}})}\|v\|_{\lambda^{k_0},D_{k_0}}\\
&\triangleq C_5(n,q,\sigma,C_0,r,B)\|v\|_{\lambda^{k_0},D_{k_0}}.
\end{aligned}$$

Here $k_0=k_0(n)$ is the smallest integer such that $\lambda^{k_0}\geq 2$. If $\beta<2$, since (29) is true, we can still do iteration. Starting from $\|v\|_{\lambda,D_1}$, in k_0 steps, we can get a control of $\|v\|_{\lambda^{k_0},D_{k_0}}$. That is,

$$\|v\|_{\lambda^{k_0},D_{k_0}} \leq C_6(n,q,\sigma,C_0,r,B)\|v\|_{\lambda,D_1}.$$

Consequently,

$$\|v\|_{\lambda^k,D_k} \le C_7(n,q,\sigma,C_0,r,B)\|v\|_{\lambda,D_1}, \quad \forall\, k \ge 0. \tag{33}$$

Actually, what we get is

$$\|v\|_{\lambda^{k_2},D_{k_2}} \le C_7(n,q,\sigma,C_0,r,B)\|v\|_{\lambda^{k_1},D_{k_1}}, \quad \forall\, 0 \le k_1 \le k_2. \tag{34}$$

From inequality (33), and $D' \subset D_k, \forall\, k \ge 0$, we get

$$\|v\|_{\lambda^k,D'} \le \|v\|_{\lambda^k,D_k} \le C_7\|v\|_{\lambda,D_1} \le C_7\|v\|_{\lambda,D}.$$

Let $k \to \infty$, $C_a \triangleq C_7(n,q,\sigma,C_0,r,B)$, we get

$$\|v\|_{\infty,D'} \le C_a(n,q,\sigma,C_0,r,B)\|v\|_{\lambda,D}\ .$$

Since $u \ge 0$, we have

$$\begin{aligned}\|u\|_{\infty,D'} &\le \|v\|_{\infty,D'}\\ &\le C_a(n,q,\sigma,C_0,r,B)\|v\|_{\lambda,D}\\ &\le C_a(n,q,\sigma,C_0,r,B)(\|u\|_{\lambda,D} + \kappa\|1\|_{\lambda,D})\\ &= C_a(n,q,\sigma,C_0,r,B)(\|u\|_{\lambda,D} + \|h\|_{q,D}\|1\|_{\lambda,D}).\end{aligned}$$

□

Remark 5.1. *From our proof, in order inequality (34) to be true, we only need* $\|f\|_{q,D_{k_1}} + \|R_-\|_{q,D_{k_1}} + 1 \le C_0$. *Consequently, inequality (25) is true for the same constant if D is replaced by D_k, i.e.,*

$$\|u\|_{\infty,D'} \le C_a(n,\sigma,C_0,r,B)(\|u\|_{\lambda^k,D_k} + \|h\|_{q,D}\|1\|_{\lambda^k,D_k}). \tag{35}$$

Lemma 5.2. $\{(M^n,g(t)),\ 0 \le t \le 1\}$ *is a closed Ricci flow solution with* $Ric \ge -B$. *There is a uniform Soblev constant σ for Ω at each time slice. If* $u \in C^1(D)$ *and* $u \ge 0$,

$$\frac{\partial u}{\partial t} \le \triangle u + fu + h,$$

in distribution sense. Here $f \in L^{\frac{n+2}{2}}(D)$. *Fix $\beta > 1$. Then there are two constants* $\delta_b(n,\sigma,\beta), C_b(n,\sigma,r,B,\beta)$ *such that if* $\|f\|_{\frac{n+2}{2},D} + \|R_-\|_{\frac{n+2}{2},D} \le \delta_b$, *then*

$$\|u\|_{\frac{n+2}{n}\beta,D_1} \le C_b(n,\sigma,r,B,\beta)(\|u\|_{\beta,D} + \|h\|_{\frac{n+2}{2},D}\|1\|_{\beta,D})\ . \tag{36}$$

Here D_1 is defined by equation (30).

Proof. Let $\eta = \eta_1$, then we do the calculation as in the proof of lemma 5.1. Instead of $\kappa = \|h\|_{q,D}$ in the previous lemma, we let $\kappa = l \cdot \|h\|_{\frac{n+2}{2},D}$ for some positive number l. We can get a similar inequality as inequality (28),

$$\|\eta_1^2 v^\beta\|_{\frac{n+2}{n},D} \le \sigma^{\frac{n}{n+2}}\Lambda(\beta)\{\int_D (|\nabla \eta_1|^2 + 2\eta_1\frac{\partial \eta_1}{\partial t})v^\beta d\mu dt +$$
$$(\|f\|_{\frac{n+2}{2},D} + \|R_-\|_{\frac{n+2}{2},D} + \frac{1}{l})\|\eta_1^2 v^\beta\|_{\frac{n+2}{n},D}\}.$$

If $\|f\|_{\frac{n+2}{2},D}+\|R_-\|_{\frac{n+2}{2},D}\leq\frac{1}{4\sigma^{\frac{n}{n+2}}\Lambda(\beta)}$, choose $l=4\sigma^{\frac{n}{n+2}}\Lambda(\beta)+1$, we obtain

$$\begin{aligned}\|\eta_1^2 v^\beta\|_{\frac{n+2}{n},D} &\leq 2\sigma^{\frac{n}{n+2}}\Lambda(\beta)\int_D(|\nabla\eta_1|^2+2\eta_1\frac{\partial\eta_1}{\partial t})v^\beta d\mu dt\\ &\leq 2\sigma^{\frac{n}{n+2}}\Lambda(\beta)C_8(r,B)\|v^\beta\|_{1,D}\ .\end{aligned}$$

Consequently,

$$\begin{aligned}\|v\|^\beta_{\frac{n+2}{n}\beta,D_1} &= \|v^\beta\|_{\frac{n+2}{n},D_1}\\ &\leq \|\eta_1^2 v^\beta\|_{\frac{n+2}{n},D}\\ &\leq 2\sigma^{\frac{n}{n+2}}\Lambda(\beta)C_8(r,B)\|v^\beta\|_{1,D}\\ &= 2\sigma^{\frac{n}{n+2}}\Lambda(\beta)C_8(r,B)\|v\|^\beta_{\beta,D}\ .\end{aligned}$$

Let $C_9(n,\sigma,r,B,\beta)\triangleq(2\sigma^{\frac{n}{n+2}}\Lambda(\beta)C_8(r,B))^{\frac{1}{\beta}}$, we get

$$\|v\|_{\frac{n+2}{n}\beta,D_1}\leq C_9(n,\sigma,r,B,\beta)\|v\|_{\beta,D}\ .$$

Since $v=u+\kappa,\ u\geq 0$,

$$\begin{aligned}\|u\|_{\frac{n+2}{n}\beta,D_1} &\leq \|v\|_{\beta\cdot\frac{n+2}{n},D_1}\\ &\leq C_9(n,\sigma,r,B,\beta)\|v\|_{\beta,D}\\ &\leq C_9(n,\sigma,r,B,\beta)(\|u\|_{\beta,D}+\|\kappa\|_{\beta,D})\\ &= C_9(n,\sigma,r,B,\beta)(\|u\|_{\beta,D}+l\cdot\|h\|_{\frac{n+2}{2},D}\|1\|_{\beta,D})\\ &\leq C_b(n,\sigma,r,B,\beta)(\|u\|_{\beta,D}+\|h\|_{\frac{n+2}{2},D}\|1\|_{\beta,D}).\end{aligned}$$

Therefore, we finish the proof if we choose

$$\begin{aligned}\delta_b(n,\sigma,\beta) &= \frac{1}{4\sigma^{\frac{n}{n+2}}\Lambda(\beta)},\\ C_b(n,\sigma,r,B,\beta) &= C_9(n,\sigma,r,B,\beta)\cdot(4\sigma^{\frac{n}{n+2}}\Lambda(\beta)+1).\end{aligned}$$

□

Before we use Moser iteration for R, we need some volume control.

Property 5.2. *$\{(M^n,g(t)),\ 0\leq t\leq 1\}$ is a closed Ricci flow solution.*

$$|Ric(x,t)|\leq(n-1),\quad\forall(x,t)\in\Omega\times[0,1].$$

Then there exists a constant $\tilde{V}(n,r)\geq 1$ such that

$$\|1\|_{q,D}\leq\tilde{V}^{\frac{1}{q}}\leq\tilde{V},\quad\forall\, q\geq 1. \tag{37}$$

Proof. Since $\Omega=B_{g(1)}(p,r)$, $Ric\leq(n-1)$, by the evolution of geodesic length under Ricci flow, we have

$$\Omega\subset B_{g(t)}(p,e^{(n-1)}r),\quad\forall t\in[0,1].$$

On the other hand, $Ric \geq -(n-1)$, by volume comparison theorem, we obtain

$$\int_{B_{g(t)}(p, e^{(n-1)}r)} d\mu \leq \alpha(n-1) \int_0^{e^{(n-1)}r} (\sinh r)^{n-1} dr \triangleq C_{10}(n,r),$$

where $\alpha(n-1)$ is the area of S^{n-1} with canonical metric. Hence,

$$\begin{aligned}
\|1\|_{1,D} &= \int_0^1 \int_\Omega d\mu dt \\
&\leq \int_0^1 \int_{B_{g(t)}(p, e^{(n-1)}r)} d\mu dt \\
&\leq C_{10}.
\end{aligned}$$

Let $\tilde{V}(n,r) \triangleq \max\{C_{10}, 1\}$, then

$$\|1\|_{q,D} = \|1\|_{1,D}^{\frac{1}{q}} \leq \tilde{V}^{\frac{1}{q}} \leq \tilde{V}, \quad \forall\, q \geq 1.$$

□

Now we can apply Moser iteration to R.

Theorem 5.1. $\{(M^n, g(t)),\ 0 \leq t \leq 1\}$ *is a closed Ricci flow solution. Suppose*

$$\begin{aligned}
&Ric(x,t) \geq -B, \quad \forall (x,t) \in M \times [0,1], \quad 0 \leq B \leq 1, \\
&Ric(x,t) \leq (n-1), \quad \forall (x,t) \in \Omega \times [0,1].
\end{aligned}$$

There is a uniform Soblev constant σ for Ω at each time slice. Then there are constants $\delta(n,\sigma,r), C(n,\sigma,r)$ such that if $\|R\|_{\frac{n+2}{2},D} + B \leq \delta$, then

$$\|R_+\|_{\infty,D'} \leq C(\|R\|_{\frac{n+2}{2},D} + B)\ . \tag{38}$$

Proof. Since $Ric \geq -B$, define $\hat{R} \triangleq R + nB$, we get inequality (7),

$$\frac{\partial \hat{R}}{\partial t} \leq \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2.$$

Because $0 \leq B \leq 1$, in $D = \Omega \times [0,1]$, $|Ric| \leq (n-1)$, by Property 5.2,

$$\|1\|_{q,D} = \|1\|_{1,D}^{\frac{1}{q}} \leq \tilde{V}^{\frac{1}{q}} \leq \tilde{V}, \quad \forall\, q \geq 1.$$

Let $u = \hat{R}, f = 2(\hat{R} - 2B), h = 2nB^2$. As in lemma 5.2, let

$$\begin{aligned}
\beta &= \frac{n+2}{2}; \\
\delta_b &= \delta_b(n,\sigma,\beta); \\
C_b &= C_b(n,\sigma,r,1,\beta).
\end{aligned}$$

If $\|R\|_{\frac{n+2}{2},D} + B$ is very small, say,

$$\|R\|_{\frac{n+2}{2},D} + B \leq \delta(n,\sigma,r) \triangleq \frac{\delta_b}{3n\tilde{V}},$$

then

$$\begin{aligned}
&\|2(\hat{R}-2B)\|_{\frac{n+2}{2},D}+\|R_-\|_{\frac{n+2}{2},D}\\
&=\|2(R+(n-2)B)\|_{\frac{n+2}{2},D}+\|R_-\|_{\frac{n+2}{2},D}\\
&\leq 3\|R\|_{\frac{n+2}{2},D}+2(n-2)B\|1\|_{\frac{n+2}{2},D}\\
&\leq 3n\tilde{V}^{\frac{2}{n+2}}(\|R\|_{\frac{n+2}{2},D}+B)\\
&\leq \frac{\delta_b}{\tilde{V}^{\frac{n}{n+2}}}\\
&\leq \delta_b,
\end{aligned}$$

hence, by lemma 5.2,

$$\begin{aligned}
\|\hat{R}\|_{\frac{n+2}{n}\frac{n+2}{2},D_1} &\leq C_b(\|\hat{R}\|_{\frac{n+2}{2},D}+2nB^2\|1\|^2_{\frac{n+2}{2},D})\\
&\leq C_b(\|R\|_{\frac{n+2}{2},D}+nB\|1\|_{\frac{n+2}{2},D}+2nB^2\|1\|^2_{\frac{n+2}{2},D})\\
&\leq C_b(\|R\|_{\frac{n+2}{2},D}+3nB\tilde{V}^{\frac{4}{n+2}})\\
&\leq C_b 3n\tilde{V}^{\frac{4}{n+2}}(\|R\|_{\frac{n+2}{2},D}+B) \qquad (39)\\
&\leq C_b\delta_b. \qquad (40)
\end{aligned}$$

Now let $q=\frac{n+2}{n}\frac{n+2}{2}>\frac{n+2}{2}$, then from inequality (40),

$$\begin{aligned}
\|2(\hat{R}-2B)\|_{q,D_1}+\|R_-\|_{q,D_1}+1 &\leq 3\|\hat{R}\|_{q,D_1}+(n+4)B\|1\|_{q,D_1}+1\\
&\leq 3C_b\delta_b+(n+4)B\tilde{V}^{\frac{1}{q}}+1\\
&\leq 3C_b\delta_b+\delta_b+1.
\end{aligned}$$

Note that $0\leq B\leq 1$, by the definition of C_a in Lemma 5.1, we get

$$C_a(n,\frac{(n+2)^2}{2n},(3C_b+1)\delta_b+1,\sigma,r,B)\leq C_a(n,\frac{(n+2)^2}{2n},(3C_b+1)\delta_b+1,\sigma,r,1).$$

Let $C_a=C_a(n,\frac{(n+2)^2}{2n},(3C_b+1)\delta_b+1,\sigma,r,1)$.

From Remark 5.1, we have

$$\begin{aligned}
\|\hat{R}\|_{\infty,D'} &\leq C_a(n,\frac{(n+2)^2}{2n},(3C_b+1)\delta_b+1,\sigma,r,B)(\|\hat{R}\|_{\frac{n+2}{n},D_1}+\|h\|_{q,D}\|1\|_{\frac{n+2}{n},D_1})\\
&\qquad \text{by Hölder inequality}\\
&\leq C_a(\|\hat{R}\|_{\frac{(n+2)^2}{2n},D_1}\|1\|_{\frac{n+2}{n},D_1}+\|2nB^2\|_{q,D}\|1\|_{\frac{n+2}{n},D_1})\\
&\qquad \text{from inequality (39)}\\
&\leq C_a(3nC_b\tilde{V}^{\frac{4}{n+2}}(\|R\|_{\frac{n+2}{2},D}+B)\|1\|_{\frac{n+2}{n},D_1}+2nB^2\|1\|_{\frac{(n+2)^2}{2n},D}\|1\|_{\frac{n+2}{n},D_1})\\
&\qquad \text{since } \tilde{V}\geq 1\\
&\leq C_a\tilde{V}^{\frac{n+4}{n+2}}(3nC_b(\|R\|_{\frac{n+2}{2},D}+B)+2nB^2)\\
&\qquad \text{note that } B^2\leq B\\
&\leq 3n(C_b+1)C_a\tilde{V}^{\frac{n+4}{n+2}}(\|R\|_{\frac{n+2}{2},D}+B). \qquad (41)
\end{aligned}$$

Note that $\|R_+\|_{\infty,D'} \leq \|\hat{R}\|_{\infty,D'}$. Let $C(n,\sigma,r) \triangleq 3n(C_b+1)C_a\tilde{V}^{\frac{n+4}{n+2}}$, from inequality (41), we have

$$\|R_+\|_{\infty,D'} \leq C(n,\sigma,r)(\|R\|_{\frac{n+2}{2},D} + B).$$

$\square$

6 Proof of Theorem 1.1 for $n \geq 3$

Proof. Since $\|R\|_{\alpha,M\times[0,T)} < \infty$ implies $\|R\|_{\frac{n+2}{2},M\times[0,T)} < \infty$ if $\alpha > \frac{n+2}{2}$, so we only need to prove Theorem 1.1 for $\alpha = \frac{n+2}{2}$. We shall argue by contradiction.

Suppose the flow cannot be extended, then $|Ric|$ is unbounded by Sesum's result. Since $Ric \geq -A$, we know

$$\sup_{(x,t)\in M\times[0,T)} R(x,t) = \infty.$$

Therefore, there exists a sequence $(x^{(i)}, t^{(i)})$ such that $\lim_{i\to\infty} t^{(i)} = T$, and

$$R(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)}]} R(x,t).$$

Consequently, $\lim_{i\to\infty} R(x^{(i)}, t^{(i)}) = \infty$. Define

$$Q^{(i)} \triangleq R(x^{(i)}, t^{(i)}), \quad P^{(i)} \triangleq B_{g(t^{(i)})}(x^{(i)}, (Q^{(i)})^{-\frac{1}{2}}) \times [t^{(i)} - (Q^{(i)})^{-1}, t^{(i)}],$$

then for any $(x,t) \in P^{(i)}$, $R(x,t) \leq Q^{(i)}$.

Now, let $g^{(i)}(t) \triangleq Q^{(i)} g((Q^{(i)})^{-1}(t-1) + t^{(i)})$. We have a sequence of Ricci flow solutions: $\{(M^n, g^{(i)}(t)),\ 0 \leq t \leq 1\}$. Moreover,

$$\begin{aligned} &R^{(i)}(x,t) \leq 1, \quad \forall\ (x,t) \in B_{g^{(i)}(1)}(x^{(i)}, 1) \times [0,1]; \\ &Ric^{(i)}(x,t) \geq -\frac{A}{Q^{(i)}}, \quad \forall\ (x,t) \in M \times [0,1]. \end{aligned} \tag{42}$$

Since $Ric^{(i)} + \frac{A}{Q^{(i)}} \geq 0$, so

$$Ric^{(i)} + \frac{A}{Q^{(i)}} \leq tr(Ric^{(i)} + \frac{A}{Q^{(i)}}) = R^{(i)} + \frac{nA}{Q^{(i)}}.$$

Consequently, $Ric^{(i)} \leq R^{(i)} + \frac{(n-1)A}{Q^{(i)}}$. Note that $\lim_{i\to\infty} \frac{A}{Q^{(i)}} = 0,\ n \geq 3$, by inequalities (42), we get

$$\begin{aligned} &Ric^{(i)}(x,t) \leq n-1, \quad \forall\ (x,t) \in B_{g^{(i)}(1)}(x^{(i)}, 1) \times [0,1]; \\ &Ric^{(i)}(x,t) \geq -\frac{A}{Q^{(i)}}, \quad \forall\ (x,t) \in M \times [0,1]. \end{aligned} \tag{43}$$

Since for any $x \in B_{g(t^{(i)})}(x^{(i)}, (Q^{(i)})^{-\frac{1}{2}})$, $-nA \leq R(x,t) \leq Q^{(i)}$, for large i, we have $|R(x,t)| \leq Q^{(i)}$. By Theorem 2.2, there exists a κ such that

$$\mathrm{Vol}_{g^{(i)}(1)}(B_{g^{(i)}(1)}(x^{(i)}, 1)) = \frac{\mathrm{Vol}_{g(t^{(i)})}(B_{g(t^{(i)}}(x^{(i)}, (Q^{(i)})^{-\frac{1}{2}}))}{(Q^{(i)})^{-\frac{n}{2}}} \geq \kappa. \tag{44}$$

From inequalities (43) and (44), we are able to use Theorem 4.1. Therefore, we get a constant $r(\kappa, n)$ such that for large i, on the geodesic ball $B_{g^{(i)}(1)}(p, r)$, there is a uniform Sobolev constant $\sigma(n, r)$ for every time slice $t \in [0, 1]$.

Then we collect conditions to use Theorem 5.1. Define

$$\Omega^{(i)} \triangleq B_{g^{(i)}(1)}(p, r), \qquad \Omega^{(i)\prime} \triangleq B_{g^{(i)}(1)}(p, \frac{r}{2}),$$
$$D^{(i)} \triangleq \Omega^{(i)} \times [0, 1], \qquad D^{(i)\prime} \triangleq \Omega^{(i)\prime} \times [\frac{1}{2}, 1].$$

Since $\int_0^T \int_M |R|^{\frac{n+2}{2}} d\mu dt$ is a scale invariant,

$$\begin{aligned}
\lim_{i\to\infty} \|R^{(i)}\|_{\frac{n+2}{2}, D^{(i)}} + \frac{A}{Q^{(i)}} &= \lim_{i\to\infty} \|R^{(i)}\|_{\frac{n+2}{2}, D^{(i)}} \\
&= \lim_{i\to\infty} \int_{t^{(i)}-(Q^{(i)})^{-1}}^{t^{(i)}} \int_{B_{g(t^{(i)})}(p, r(Q^{(i)})^{-\frac{1}{2}})} |R|^{\frac{n+2}{2}} d\mu dt \\
&\leq \lim_{i\to\infty} \int_{t^{(i)}-(Q^{(i)})^{-1}}^{t^{(i)}} \int_M |R|^{\frac{n+2}{2}} d\mu dt \\
&= 0.
\end{aligned}$$

The last step comes from $\int_0^T \int_M |R|^{\frac{n+2}{2}} d\mu dt < \infty$ and $\lim_{i\to\infty}(Q^{(i)})^{-1} = 0$. Consequently, for large i, $\|R^{(i)}\|_{\frac{n+2}{2}, D^{(i)}} + \frac{A}{Q^{(i)}} \leq \delta(n, \sigma, r)$. From Theorem 5.1, we know

$$\|R_+^{(i)}\|_{\infty, D'} \leq C(n, \sigma, r)(\|R^{(i)}\|_{\frac{n+2}{2}, D^{(i)}} + \frac{A}{Q^{(i)}}) \; . \tag{45}$$

Taking limit on both sides, we get

$$\lim_{i\to\infty} \|R_+^{(i)}\|_{\infty, D'} \leq \lim_{i\to\infty} C(n, \sigma, r)(\|R^{(i)}\|_{\frac{n+2}{2}, D^{(i)}} + \frac{A}{Q^{(i)}}) = 0.$$

On the other hand,

$$\lim_{i\to\infty} \|R_+^{(i)}\|_{\infty, D'} \geq \lim_{i\to\infty} R_+^{(i)}(x^{(i)}, 1) = 1.$$

Therefore we get a contradiction. □

Remark 6.1. *From the proof, we know the condition $Ric \geq -A$ is used only to assure that after blowup, Ricci curvature becomes almost nonnegative. However, when* $\dim = 3$*, this can be achieved automatically. Actually, by Hamilton-Ivey's pinch[cf.(11), Theorem4.1],*

$$R \geq |\nu|(\log |\nu| + \log(1 + t) - 3).$$

Here $\nu(x, t)$ is the smallest eigenvalue of the curvature operator and we have normalized the initial metric such that $inf_{x\in M}\nu(x, 0) \geq -1$. This tells us that Ricci curvature must be nonnegative after blowup. Therefore, we can get the following Corollary.

Corollary 6.1. $\{(M^3, g(t)), 0 \leq t < T < \infty\}$ *is a closed Ricci flow solution. If* $\|R\|_{\alpha,\, M\times[0,T)} < \infty$, $\alpha \geq \frac{5}{2}$*, then this flow can be extended over time T.*

A natural question is whether the Ricci lower bound condition superfluous in higher dimension. To be conservative, can we substitute the condition $Ric \geq -A$ by a weaker one?

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